Part 2: Question 1 (20 points)

Consider a pure exchange economy with L goods, I consumers and one firm whose production technology is free disposal $Y_1 = -\mathbb{R}^L_+$. Each consumer i's preferences $\succeq_i$ satisfy completeness, transitivity, continuity, strict convexity, and strong monotonicity. The consumers' consumption sets are $X_i = \mathbb{R}^L_+$. The aggregate endowment vector satisfies $\bar{\boldsymbol{\omega}} \gg \mathbf{0}$ and each consumer owns an equal share in the firm.

In class we proved that for an economy satisfying these properties, if we allocated the initial endowment such that $\boldsymbol{\omega}_i \gg \mathbf{0}$ for all i, it is guaranteed to have an equilibrium. You may use this fact below without proving it.

(i) [**7 points:**] Show that an allocation $(\boldsymbol{x}, \boldsymbol{y})$ satisfying $\boldsymbol{x}_i \succeq_i \boldsymbol{x}_j$ for all $i, j \in \{1, \dots, I\}$[1], does not necessarily imply the allocation is Pareto optimal.

(ii) [**6 points:**] If the initial endowment was allocated such that $\boldsymbol{\omega}_i \gg \mathbf{0}$ for all i, we know from above that an equilibrium is guaranteed to exist. Denote the allocation and price vector in equilibrium as $(\boldsymbol{x}^\star, \boldsymbol{y}^\star, \boldsymbol{p})$. Prove that this equilibrium allocation is Pareto optimal.

(iii) [**7 points:**] Prove that an allocation $(\boldsymbol{x}, \boldsymbol{y})$ that satisfies both Pareto optimality and $\boldsymbol{x}_i \succeq_i \boldsymbol{x}_j$ for all $i, j \in \{1, \dots, I\}$ exists in this economy.

Part 2: Question 2 (20 points)

Consider an economy with $L = 3$ goods. There is one consumer with the utility function:

$$u = v_1(x_1) + v_2(x_2) + x_3$$

where $v'_\ell(x_\ell) > 0$ and $v''_\ell(x_\ell) < 0$ for all $x_\ell \geq 0$ and $\ell = 1, 2$. The consumer must consume nonnegative quantities of goods 1 and 2 but can consume any quantity of good 3. The consumption set is therefore $X = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$. The consumer has an endowment $\omega_3 > 0$ of good 3 and none of goods 1 and 2 ($\omega_1 = \omega_2 = 0$). The prices of goods 1 and 2 are p_1 and p_2 and the price of good 3 is normalized to 1.

There are two price-taking firms owned by the single consumer whose production sets are:

$$\mathcal{Y}_1 = \left\{\boldsymbol{y}_1 \in \mathbb{R}^3 : y_{11} \leq f_1(-y_{31}), y_{21} = 0, y_{31} \leq 0\right\}$$
$$\mathcal{Y}_2 = \left\{\boldsymbol{y}_2 \in \mathbb{R}^3 : y_{12} = 0, y_{22} \leq f_2(-y_{32} - e(y_{11})), y_{32} + e(y_{11}) \leq 0\right\}$$

[1] In words: in the allocation, no consumer strictly prefers another consumer's bundle to their own.

where $f_j'(\cdot) < 0$ and $f_j''(\cdot) > 0$ for $j = 1, 2$ for all $y_{3j} \leq 0$. This means that a higher input $-y_{3j}$ leads to more output, but at a decreasing rate. Notice that firm 1 has a negative production externality on firm 2. It essentially acts as a fixed cost for firm 2. Firm 2 needs to use $e(y_{11})$ units of the input good 3 to "pay for" the externality before it can use additional units of the input to product good 2. The externality satisfies $e'(y_{11}) > 0$ for all $y_{11} \geq 0$.

Define the following functions:

$$c_1(y_{11}) = \min_{-y_{31} \geq 0} \{-y_{31} \text{ subject to } f_1(-y_{31}) \geq y_{11}\}$$

$$c_2(y_{22}) = \min_{-y_{32} \geq e(y_{11})} \{-y_{32} \text{ subject to } f_2(-y_{32} - e(y_{11})) \geq y_{22}\}$$

where $c_j'(\cdot) > 0$ and $c_j''(\cdot) > 0$. Putting these together we can write the firms' profit functions as

$$\pi_1(p_1) = \max_{y_{11} \geq 0} \{p_1 y_{11} - c_1(y_{11})\}$$

$$\pi_2(p_2) = \max_{y_{22} \geq 0} \{p_2 y_{22} - c_2(y_{22}) - e(y_{11})\}$$

Because the consumer owns both firms, the consumer's wealth is given by $\omega_3 + \pi_1(p_1) + \pi_2(p_2)$.

Assume that $v_\ell'(0) > c_\ell'(0)$ for $\ell = 1, 2$ and that $v_1'(0) > c_1'(0) + e_1'(0)$ such that all solutions are interior. You do not need to consider boundary cases in your solution.

(i) [**5 points**] Characterize the market equilibrium in this economy, assuming that each firm maximizes its own profits (i.e. the single consumer that owns the firms does not dictate their production, but only collects the profits at the end).

(ii) [**5 points**] Characterize the Pareto optimal level of output of each firm assuming the social planner seeks to maximize the utility of the single consumer. Show that this is different to part (i).

(iii) [**5 points**] Suppose there is a government that can impose a per-unit tax on the production of good 1. What should the tax be set at in order to achieve the Pareto optimal solution in part (ii)?

(iv) [**5 points**] Suppose now the government did not know the cost functions, nor the cost of the externality. They instead passed the following law. Before production takes place, both firms are required to publicly announce a suggested per-unit tax rate $t_j \geq 0$, $j = 1, 2$, on the output of good 1. Then, after production takes place, the following occurs:

- Firm 1 pays the tax suggested by firm 2 (i.e. $t_2 x_1$) to the government.
- Firm 2 receives compensation from the government based on the tax rate suggested by firm 1 (i.e. $t_1 x_1$).
- Both firms are required to pay a tax equal to $(t_2 - t_1)^2$ to the government.

Show that it is optimal for both firms to suggest the same tax, and this tax equals the one from part (iii).

Suggested steps: Think of this as a sequential game with 2 stages and solve it with backward induction. In stage 2, firms choose the optimal production levels given the tax rates from stage 1. In stage 1, firms choose the optimal tax rates knowing that they will optimize given those tax rates in stage 2 (i.e. firm 2 takes into account that it can impact y_{11} through t_2).